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# Spectral Properties of the Second Order Differential Operators with Eigenvalues Parameter Dependent Boundary Conditions

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**Abstract**

We consider the second-order spectral problem $-y''(x) + q(x)y(x) = \lambda^2\rho(x)y(x)$, $x \in (0,1)$ with spectral parameter in the boundary condition. We determine the linear independent solutions (eigenfunctions) for the given spectral problem. Further we investigate locations (in complex plane) and asymptotic behaviour of the eigenvalues for regular and irregular cases of this problem.

## Introduction

The Boundary value problems for ordinary differential operators with spectral parameter in the boundary conditions have been considered in various formulations by many authors (see $[4-5,7\,,13-18]$). In [11, 13], the authors studied the basis property in various function spaces of the eigenvalues- and associated function system of the Sturm-Liouville spectral problem with spectral parameter in the boundary conditions. The existence of eigenvalues, estimates of eigenvalues and eigenfunctions, oscillation properties of eigenfunctions and expansion theorems were considered in [5, 6, 15, 17] for fourth-order ordinary differential operators with a spectral parameter in a boundary condition. In [12], Menken and Mamedov have studied the non-self adjoint Sturm-Liouville operators with periodic and anti-periodic boundary conditions are and they proved basis property in the space $L_p(0,1)$ of the root functions. In [8], Mamedov investigated the completeness, minimality and basic properties of the eigenfunctions of one discontinuous Sturm-Liouville problem with a spectral parameter in boundary conditions and transmission conditions. In [9], Mamedov and Menken are studied the boundary value problem generated by a second order differential equation and a spectral parameter dependent boundary condition on the half line. They are defined the scattering data and they proved the continuity of the scattering function S(λ) with deriving the Levinson-type formula. In [10], Mamedov and Cetinkaya are considered the Sturm-Liouville operator with discontinuous coefficient and a spectral parameter in boundary conditions also they are investigated orthogonality of the eigenfunctions, realness and simplicity of the eigenvalues. Moreover they have shown that the eigenfunctions form a complete system and expansion formula with respect to eigenfunctions is obtained. In [13], Naimark investigated the spectrum and the expansion in eigenfunctions of a non-self adjoint operator of second order on a semi-axis. In [4], Fulton considered the Sturm-Liouville equation with one boundary condition dependent on the spectral parameter and obtained asymptotic estimates of

eigenvalues or eigenfunctions.
In [7], Jwamer and Mohammed studied the boundedness of the eigenfunctions of the spectral problem of the form:
$-y''(x) + p_1(x)y'(x) + q_1(x)y(x) = \lambda^2\rho(x)y(x),$
with the boundary conditions
$y(0) = 0, y'(a) - i\lambda y(a) = 0,$
$$\left(\int_0^a \frac{\rho(x)}{e^{\int p_1(x)dx}} |y(x)|^2\, dx\right)^{\frac{1}{2}} = 1,$$
where $\lambda$ is a spectral parameter, $\rho(x)$ is a weight function satisfying the Lipschitz condition, $p_1(x) \neq 0, p_1(x) \in C^1[0,a]$ and $q_1(x) \in C[0,a]$.
In [1-3], the properties of eigenvalues and the estimation of the corresponding eigenfunctions for the boundary value problems consisting the same differential equation
$-y'' + q(x)y = \lambda^2\rho(x)y, x \in (0,a)$
with different boundary conditions were studied.
In this paper, we concern to find the liner independent solutions (eigenfunctions) as well as the asymptotic formulas for eigenvalues in both regular and irregular cases of the following spectral boundary value problem:

$$-y''(x) + q(x)y(x) = \lambda^2\, \rho(x)y(x), x \in (0,1) \qquad (1)$$
$$U_1(y) = y'(0) - i\lambda h y(0) = 0, \qquad (2)$$
$$U_2(y) = y'(1) + i\lambda h y(1) = 0, \qquad (3)$$

where $h$ is a positive real parameter, $\lambda = \theta + i\gamma$ is an eigenvalue, where $i = \sqrt{-1}, \theta, \gamma \in \mathbb{R}$, and $\mathbb{R}$ is the set of all real numbers and $\rho(x), q(x)$ are positive real continuous functions on (0,1), where $\rho(x)$ is known as a weight function. Let n and N be fixed constants such that $0 < n \leq N$ and $L^+(0,1)$ refers to the set of all positive integrable functions $\rho(x)$ on interval (0,1)that satisfy the condition $n \leq \rho(x) \leq N$. We divide $\lambda$ −complex plane into four sectors $T_k\ and\ \overline{T_k}$ , $k = 0,1,2,3,$ defined by $\frac{k\pi}{2} \leq \arg\lambda \leq \frac{k\pi}{2} + \frac{\pi}{4}$. We denote the two different roots of (1) by $w_o, w_1$ arranged so that for each of the sectors $T_k$ can be ordered in such a way that, for all $\lambda \in T_k$, the inequality $Re(iw_o\lambda) \leq Re(iw_1\lambda)$ hold, where $Re(iw_j\lambda)$ means the real part of $iw_j\lambda , j = o, 1$.
The paper is arranged as follows: In section 2, we find linear independent solutions of differential problem (1). In Section 3, we specify the position of the spectral parameter in complex plane. In Section 4, we obtain new accurate asymptotic formulas of the eigenvalues for regular and irregular cases.

## 2. The Behaviour of the Solution of the Second Order Boundary Value Problem (1)-(3)

The aim of this section is to estimate the behaviour of the solutions to the given second order boundary value problem and finding its coefficients, $A_o(x)$, from the following theorem

### Theorem 1

If the functions $\rho(x)$ and $q(x)$ are continuous in the interval (0,1), then the equation (1) has, for each region $T$ of the $\lambda -$ complex−plane, two linearly independent solutions $y_1$and $y_2$ which are regular for $\lambda \in T$. For sufficiently large $|\lambda|$, they can be expressed in the form
$$y_k(x,\lambda) = e^{\lambda\int_0^x \emptyset_k(t)\, dt}\left(\frac{1}{\sqrt[4]{\rho(x)}} + O\left(\frac{1}{\lambda}\right)\right),$$
where $\emptyset_k(t) = iw_k\sqrt{\rho(t)}, for\ k = 1,2.$

### Proof

From [19], Tamarkin has proved that the linear independent solutions of equation (1), for sufficient large $|\lambda|$, can be written in the form

$$y_k(x,\lambda) = e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x) + O\left(\frac{1}{\lambda}\right)\right), k = 1,2,$$

where $A_o(x)$ is an unknown function of $x$ to be determined.

By differentiating this equation up to second order with respect to $x$, the following relations are obtained

$$y_k'(x,\lambda) = \left(\lambda\,\emptyset_k(x)\right)e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x) + \frac{1}{\lambda}\left(\frac{A_o'(x)}{\emptyset_k(x)}\right) + O\left(\frac{1}{\lambda}\right)\right),$$

$$y_k''(x,\lambda) = \left(\lambda\,\emptyset_k(x)\right)^2 e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x) + \frac{1}{\lambda}\left(\frac{2A_o'(x)}{\emptyset_k(x)} + \frac{\emptyset_k'(x)}{\left(\emptyset_k(x)\right)^2}A_o(x)\right) + \frac{1}{\lambda^2}\left(\frac{A_o''(x)}{\left(\emptyset_k(x)\right)^2}\right)\right.$$
$$\left. + O\left(\frac{1}{\lambda}\right)\right).$$

By substituting $y_k(x,\lambda), y_k'(x,\lambda)$ and $y_k''(x,\lambda)$ in equation (1), we have

$$-\left(\lambda\,\emptyset_k(x)\right)^2 e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x) + \frac{1}{\lambda}\left(\frac{2A_o'(x)}{\emptyset_k(x)} + \frac{\emptyset_k'(x)}{\left(\emptyset_k(x)\right)^2}A_o(x)\right) + \frac{1}{\lambda^2}\left(\frac{A_o''(x)}{\left(\emptyset_k(x)\right)^2}\right)\right)$$
$$+ q(x)e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x)\right) + O\left(\frac{1}{\lambda}\right) = \lambda^2\rho(x)\,e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(A_o(x)\right) + O\left(\frac{1}{\lambda}\right),$$

$$\lambda^2\left(-\left(\emptyset_k(x)\right)^2 A_o(x)\right) + \lambda\left(-2\emptyset_k(x)A_o'(x) - \emptyset_k'(x)A_o(x)\right) + \left(q(x)A_o(x) - A_o''(x)\right) + O\left(\frac{1}{\lambda}\right)$$
$$= \lambda^2\rho(x)A_o(x) + O\left(\frac{1}{\lambda}\right).$$

Equating similar terms in both sides gives

$$-\left(\emptyset_k(x)\right)^2 A_o(x) = \rho(x)A_o(x),$$
$$-2\emptyset_k(x)A_o'(x) - \emptyset_k'(x)A_o(x) = 0$$

and

$$q(x)A_o(x) - A_o''(x) = 0.$$

From $-\left(\emptyset_k(x)\right)^2 A_o(x) = \rho(x)A_o(x)$, we get

$\emptyset_k(x) = \mp i\sqrt{\rho(x)}$, where $i = \sqrt{-1}$

and the equation $-2\emptyset_k(x)A_o'(x) - \emptyset_k'(x)A_o(x) = 0$,

implies that

$$2\frac{A_o'(x)}{A_o(x)} = -\frac{\emptyset_k'(x)}{\emptyset_k(x)}, A_o(x) = \frac{1}{\sqrt[4]{\rho(x)}}.$$

Finally, from equation $q(x)A_o(x) - A_o''(x) = 0$, we deduce that

$$q(x) = \frac{A_o''(x)}{A_o(x)} = \frac{5}{16}\frac{\left(\rho'(x)\right)^2}{\rho^2(x)} - \frac{1}{4}\frac{\rho''(x)}{\rho(x)}.$$

Hence,

$$y_k(x,\lambda) = e^{\lambda\int_0^x \emptyset_k(t)\,dt}\left(\frac{1}{\sqrt[4]{\rho(x)}} + O\left(\frac{1}{\lambda}\right)\right).$$

The proof of Theorem 1 is completed.

## 3. Location of Spectral Parameter

In this section, we determine the position of eigenvalues in the complex plane.

**Theorem 2**

Let $\lambda$ be an eigenvalue corresponding to the eigenfunction $y(x)$ of the problem (1)-(3). If $\theta \neq 0$, then $\lambda$ is complex and located in upper half plane.

**Proof**

In equation (1) and boundary conditions (2) and (3), we replace $y(x)$ by $\bar{y}(x)$, then the following equations are obtained:

$$-\bar{y}''(x) + q(x)\bar{y}(x) = \bar{\lambda}^{\,2}\rho(x)\bar{y}(x), x \in (0,1), \tag{4}$$
$$\bar{y}'(0) + i\bar{\lambda}\, h\, \bar{y}(0) = 0, \tag{5}$$
$$\bar{y}'(1) - i\bar{\lambda}\, h\, \bar{y}(1) = 0, \tag{6}$$

the above equations (4), (5) and (6) exhibit that $\bar{y}(x)$ is the eigenfunction corresponding to the eigenvalue $\bar{\lambda} = \theta - i\gamma$.

Multiply equation (1) by $\bar{y}(x)$ and integrate the resulting equation with respect to $x$ from 0 up to 1, we have

$$-[\bar{y}(x)y'(x)]_0^1 + \int_0^1 |y'(x)|^2\,dx + \int_0^1 q(x)\,|y(x)|^2 dx = \lambda^2 \int_0^1 \rho(x)\,|y(x)|^2 dx\,.$$

Due to boundary conditions (2) and (3), we have

$$ih\lambda|y(1)|^2 + ih\lambda|y(0)|^2 + \int_0^1 |y'(x)|^2\,dx + \int_0^1 q(x)\,|y(x)|^2 dx$$
$$= \lambda^2 \int_0^1 \rho(x)\,|y(x)|^2 dx\,. \tag{7}$$

Multiplying equation (4) by $y(x)$ and integrating the resulting equation with respect to $x$ from 0 to 1, yields

$$-[\,y(x)\,\bar{y}'(x)]_0^1 + \int_0^1 |y'(x)|^2\,dx + \int_0^1 q(x)\,|y(x)|^2 dx = \bar{\lambda}^{\,2} \int_0^1 \rho(x)\,|y(x)|^2 dx\,.$$

Through using the boundary conditions (5) and (6), we obtain

$$-ih\bar{\lambda}|y(1)|^2 - ih\bar{\lambda}|y(0)|^2 + \int_0^1 |y'(x)|^2\,dx + \int_0^1 q(x)\,|y(x)|^2 dx$$
$$= \bar{\lambda}^{\,2} \int_0^1 \rho(x)\,|y(x)|^2 dx\,. \tag{8}$$

Subtracting equation (8) from equation (7) we deduce

$$ih\,(\lambda + \bar{\lambda})(|y(1)|^2 + |y(0)|^2) = (\lambda - \bar{\lambda})(\lambda + \bar{\lambda}) \int_0^1 \rho(x)\,|y(x)|^2 dx\,.$$

Obviously, $(\lambda + \bar{\lambda}) \neq 0$ because $\theta \neq 0$, then

$$h\,(|y(1)|^2 + |y(0)|^2) = 2\gamma \int_0^1 \rho(x)\,|y(x)|^2 dx,$$

thus,

$$\gamma = \frac{h\,(|y(1)|^2 + |y(0)|^2)}{2\int_0^1 \rho(x)\,|y(x)|^2 dx}\,,$$

Since $\int_0^1 \rho(x)\,|y(x)|^2 dx$ has a positive value in the interval $(0,1)$ and $h$ is a positive real parameter, then $\gamma > 0$.

Hence, $\lambda$ is complex and located in upper half plane. Thus, the theorem is proved.

## 4. Asymptotic Behaviours of Eigenvalues to the Problem (1)-(3)

The purpose of this section is to study the asymptotic behaviours of eigenvalues in regular and irregular cases.

### Theorem 3

In the case of regular $(\rho(1) \neq 1)$, the eigenvalues of problem (1)-(3) has the following asymptotic behaviour:

1. $\lambda = \frac{1}{b}\left(m\pi - \frac{i}{2}\, Lnf(\lambda) + O\left(\frac{1}{\lambda}\right)\right), m = 0, \mp 1, \mp 2, \dots,$ in the sector $T_1$.
2. $\lambda = \frac{1}{b}\left(m\pi + \frac{i}{2}\, Lnf(\lambda) + O\left(\frac{1}{\lambda}\right)\right), m = 0, \mp 1, \mp 2, \dots,$

where $f(\lambda) = \dfrac{(i\lambda k + a)(i\lambda k_3 + a_1)}{(i\lambda k_2 + a)(i\lambda k_1 + a_1)}$ , k, $k_1, k_2, k_3, a,$ and $a_1$ are constants, in the sector $T_2$.

### Proof

Consider the determinant of $\Delta(\lambda)$, defined by $\Delta(\lambda) = \left|U_j(y_k)\right|_{k,j=1,2.}$, where

$$y_k(x,\lambda) = e^{\lambda \int_0^x \emptyset_k(t)\, dt}\left(\frac{1}{\sqrt[4]{\rho(x)}} + O\left(\frac{1}{\lambda}\right)\right),$$

and

$$y_k'(x,\lambda) = \left(\lambda\, \emptyset_k(x)\right) e^{\lambda \int_0^x \emptyset_k(t)\, dt}\left(\frac{1}{\sqrt[4]{\rho(x)}} + O\left(\frac{1}{\lambda}\right)\right) + e^{\lambda \int_0^x \emptyset_k(t)\, dt}\left(-\frac{\rho'(x)}{4\sqrt[4]{\rho^5(x)}} + O\left(\frac{1}{\lambda}\right)\right), k = 1,2.$$

The following results can be obtained by using $y_k(x,\lambda), y_k'(x,\lambda)$ and the boundary conditions (2) and (3)

$$i\lambda \frac{1}{\sqrt[4]{\rho(0)}}\,[1]\left[\sqrt{\rho(0)} - h\right] - \frac{\rho'(0)}{4\rho^{\frac{5}{4}}(0)}[1] = 0,$$

$$i\lambda e^{i\lambda b} \frac{1}{\sqrt[4]{\rho(1)}}\,[1]\left[\sqrt{\rho(1)} + h\right] - e^{i\lambda b}\frac{\rho'(1)}{4\rho^{\frac{5}{4}}(1)}[1] = 0,$$

$$-i\lambda \frac{1}{\sqrt[4]{\rho(0)}}\,[1]\left[\sqrt{\rho(0)} + h\right] - \frac{\rho'(0)}{4\rho^{\frac{5}{4}}(0)}[1] = 0,$$

and

$$i\lambda e^{-i\lambda b} \frac{1}{\sqrt[4]{\rho(1)}}\,[1]\left[h - \sqrt{\rho(1)}\right] - e^{-i\lambda b}\frac{\rho'(1)}{4\rho^{\frac{5}{4}}(1)}[1] = 0.$$

Let $\frac{1}{\sqrt[4]{\rho(0)}}\left[\sqrt{\rho(0)} - h\right] = k, \frac{1}{\sqrt[4]{\rho(1)}}\left[\sqrt{\rho(1)} + h\right] = k_1, -\frac{1}{\sqrt[4]{\rho(0)}}\left[\sqrt{\rho(0)} + h\right] = k_2, b = \int_0^1 \sqrt{\rho(t)}\, dt,$

$$\frac{1}{\sqrt[4]{\rho(1)}}\left[h - \sqrt{\rho(1)}\right] = k_3, -\frac{\rho'(0)}{4\rho^{\frac{5}{4}}(0)} = a, -\frac{\rho'(1)}{4\rho^{\frac{5}{4}}(1)} = a_1,$$

and

$$[1] = \left[1 + O\left(\frac{1}{\lambda}\right)\right].$$

So the above equations reduces to

$i\lambda k[1] + a[1] = 0,$

$i\lambda k_1 e^{i\lambda b}\,[1] + a_1\, e^{i\lambda b}\,[1] = 0,$

$i\lambda k_2\,[1] + a\,[1] = 0,$

$i\lambda k_3 e^{-i\lambda b}\,[1] + a_1\, e^{-i\lambda b}\,[1] = 0.$

Substituting these equations into the determinant $\Delta(\lambda)$, and setting $\Delta(\lambda) = 0$, we have

$$\Big((i\lambda k + a)(i\lambda k_3 + a_1) - (i\lambda k_2 + a)(i\lambda k_1 + a_1)\, e^{i2\lambda b}\Big)[1] = 0,$$

$$e^{i2\lambda b} = \frac{(i\lambda k + a)(i\lambda k_3 + a_1)}{(i\lambda k_2 + a)(i\lambda k_1 + a_1)},$$

then,

$$e^{i2\lambda b} = f(\lambda).$$

Thus,

$$i2\lambda b = Ln\, f(\lambda) + 2m\pi i + O\left(\frac{1}{\lambda}\right),$$

$$\lambda = \frac{1}{b}\left(m\pi - \frac{i}{2}\, Ln\, f(\lambda) + O\left(\frac{1}{\lambda}\right)\right).m = 0, \mp 1, \mp 2, \ldots$$

Hence, in the case of regular and in the sector $T_1$ asymptotic behaviour of spectrum has the form

$$\lambda = \frac{1}{b}\left(m\pi - \frac{i}{2}\, Ln\, f(\lambda) + O\left(\frac{1}{\lambda}\right)\right), m = 0, \mp 1, \mp 2, \ldots$$

and in the sector $T_2$ has the form

$$\lambda = \frac{1}{b}\left(m\pi + \frac{i}{2}\, Ln\, f(\lambda) + O\left(\frac{1}{\lambda}\right)\right).m = 0, \mp 1, \mp 2, \ldots$$

Thus the proof of Theorem 3 is finished.

**Theorem 4**

In the case of irregular $(\rho(1) = 1)$, the eigenvalues of problem (1)-(3) has the following asymptotic behaviour:

1. $\lambda = \frac{1}{b}\left(k\pi - \frac{i}{2}\, Lng(\lambda) + O\left(\frac{1}{\lambda}\right)\right), k = 0, \mp 1, \mp 2, \ldots,$ in the sector $T_1$.
2. $\lambda = \frac{1}{b}\left(k\pi + \frac{i}{2}\, Lng(\lambda) + O\left(\frac{1}{\lambda}\right)\right), k = 0, \mp 1, \mp 2, \ldots$

where $g(\lambda) = \dfrac{s_4(i\lambda s_1 + a)}{s_2(i\lambda s_3 + a)}$ , $s_1, s_2, s_3, s_4$ and $a$ are constants, such that $h \neq -1$, in the sector $T_2$.

**Proof**

By the same way and what we have done in Theorem 3, the following equations are obtained:

$$i\lambda \frac{1}{\sqrt[4]{\rho(0)}}\,[1]\left[\sqrt{\rho(0)} - h\right] - \frac{\rho'(0)}{4\rho^{\frac{5}{4}}(0)}[1] = 0,$$

$$i\lambda e^{i\lambda b} \frac{1}{\sqrt[4]{\rho(1)}}\,[1]\left[\sqrt{\rho(1)} + h\right] - e^{i\lambda b}\frac{\rho'(1)}{4\rho^{\frac{5}{4}}(1)}[1] = 0,$$

$$-i\lambda \frac{1}{\sqrt[4]{\rho(0)}}\,[1]\left[\sqrt{\rho(0)} + h\right] - \frac{\rho'(0)}{4\rho^{\frac{5}{4}}(0)}[1] = 0,$$

and

$$i\lambda e^{-i\lambda b} \frac{1}{\sqrt[4]{\rho(1)}}\,[1]\left[h - \sqrt{\rho(1)}\right] - e^{-i\lambda b}\frac{\rho'(1)}{4\rho^{\frac{5}{4}}(1)}[1] = 0.$$

Since in the irregular case the value of $\rho(x)$ at $x = 1$ is a constant and is equal to 1 $(i.e.;\ \rho(1) = 1)$, then $\rho'(x) = 0$ at $x = 1$, $(i.e.;\ \rho'(1) = 0)$. Thus the above equations become

$i\lambda s_1[1] + a[1] = 0,$

$i\lambda s_2 e^{i\lambda b}\,[1] = 0,$

$i\lambda s_3\,[1] + a\,[1] = 0,$

and
$i\lambda s_4 e^{-i\lambda b}\,[1] = 0,$
where
$\frac{1}{\sqrt[4]{\rho(0)}}\left[\sqrt{\rho(0)} - h\right] = s_1, 1 + h = s_2,$
$-\frac{1}{\sqrt[4]{\rho(0)}}\left[\sqrt{\rho(0)} + h\right] = s_3, h - 1 = s_4.$
Spectrum of the problem (1)-(3) coincides with the set of roots of the equation $\Delta(\lambda) = 0$, then

$$\Delta(\lambda) = \begin{vmatrix} i\lambda s_1[1] + a[1] & i\lambda s_3\,[1] + a\,[1] \\ i\lambda s_2 e^{i\lambda b}\,[1] & i\lambda s_4 e^{-i\lambda b}\,[1] \end{vmatrix} = 0.$$

After expanding the determinant and simplification, we obtain
$(i\lambda s_1 + a)s_4\, e^{-i\lambda b} - (i\lambda s_3 + a)s_2\, e^{i\lambda b} = 0,$
$e^{i2\lambda b} = g(\lambda).$
Therefore,
$i2\lambda b = Ln\, g(\lambda) + 2k\pi i + O\left(\frac{1}{\lambda}\right),$
$\lambda = \frac{1}{b}\left(k\pi - \frac{i}{2}\, Ln\, g(\lambda) + O\left(\frac{1}{\lambda}\right)\right). k = 0, \mp 1, \mp 2, \ldots$
Thus, in the case of irregular, the asymptotic behaviour of spectrum in sector $T_1$ has the form
$\lambda = \frac{1}{b}\left(k\pi - \frac{i}{2}\, Ln\, g(\lambda) + O\left(\frac{1}{\lambda}\right)\right), k = 0, \mp 1, \mp 2, \ldots$
and in sector $T_2$ has the form
$\lambda = \frac{1}{b}\left(k\pi + \frac{i}{2}\, Ln\, g(\lambda) + O\left(\frac{1}{\lambda}\right)\right). k = 0, \mp 1, \mp 2, \ldots$
Hence, Theorem 4 is proved.

## 5. Conclusions

The linear independent solutions as well as the location of the spectral parameter and asymptotic formula of eigenvalues in regular and irregular cases have been specified of the defined problem (1)-(3),

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